J. Appl. Math. Comput. (2016) 50:199–225
DOI 10.1007/s12190-015-0866-x

ORIGINAL RESEARCH

An iterative technique for solving singularly perturbed parabolic PDE

M. P. Rajan · G. D. Reddy

Received: 1 December 2014 / Published online: 18 January 2015

Abstract In this paper, we examine the applicability of a variant of iterative Tikhonov regularization for solving parabolic PDE with its highest order space derivative multiplied by a small parameter ϵ. The solution of the operator equation $\frac{\partial u}{\partial t} - \epsilon \frac{\partial^2 u}{\partial x^2} + a(x,t) = f(x,t)$ is not uniformly convergent to the solution of the operator equation $\frac{\partial u}{\partial t} + a(x,t) = f(x,t)$, when $\epsilon \to 0$. Although many numerical techniques are employed in practice to tackle the problem, the discretization of the PDE often leads to ill-conditioned system and hence the perturbed parabolic operator equation become ill-posed. Since we are dealing with unbounded operators, first we discuss the general theory for unbounded operators for iterated regularization scheme and propose an a posteriori parameter choice rule for choosing a regularization parameter in the iterative scheme. We then apply these techniques in the context of perturbed parabolic problems. Finally, we implement our iterative scheme and compare with other basic existing schemes to assert the adaptability of the scheme as an alternate approach for solving the problem.

Keywords Singular Perturbations · Parabolic PDE · Regularization

Mathematical Subject Classification 65M60 · 65M15 · 65M12

M. P. Rajan (✉) · G. D. Reddy
School of Mathematics, Indian Institute of Science Education and Research Thiruvananthapuram, CET Campus, Thiruvananthapuram 695 016, Kerala, India
e-mail: rajanmp@iisertvm.ac.in

G. D. Reddy
e-mail: damu@iisertvm.ac.in

1 Introduction

Many phenomena in science and engineering, for instance in chemical reactions, are models of reaction-diffusion singularly perturbed problems, for which the diffusion coefficient can be very small with respect to the reaction coefficient. We consider the following singularly perturbed 1D parabolic reaction-diffusion problem:

$$\frac{\partial u}{\partial t} - \epsilon \frac{\partial^2 u}{\partial x^2} + a(x,t)u = f(x,t), \;\; \Omega := (0,1) \times (0,1), \tag{1.1}$$

$$u(x,0) = 0, \;\; u(0,t) = u(1,t) = 0, \;\; t, \; x \in [0,1], \tag{1.2}$$

where $0 < \epsilon << 1$ and the reaction term satisfies $0 < \beta \leq a(x,t)$ for all $(x,t) \in \bar{\Omega}$ and data f are sufficiently smooth functions on $\bar{\Omega}$. It is well known that the solution of (1.1) and (1.2), exhibit a boundary layer at $x = 0$ and $x = 1$ [8,13,16]. The singularly perturbed time-dependent problems have been an interesting subject for many researchers and many numerical methods were proposed in literature to solve parabolic problems [1–3,8–13,16,18,19]. Although various numerical schemes are available in literature to find the approximate solution of such problems, the discretization of the PDE often leads to a highly ill-conditioned system and that results in an unstable solution. In addition to this, the presence of boundary layers phenomena prompt us to seek more stable and robust numerical methods that give a stable solution for any value of the diffusion parameter. Instead of using uniform meshes for discreteizing the problem, researchers exploited non-uniform meshes such as Shishkin Scheme and other hybrid schemes for solving the problem. In this paper, our idea is to propose iterated regularization technique as an alternate method to solve this problem. Regularization techniques have been studied in literature [4,6,7,14,15] to solve ill-posed operator equations and Tikhonov regularization is one such schemes. Our intention is to consider iterated Tikhonov regularization as an alternate tool for solving this problem and Tikhonov regularization is a special case of such a scheme. Iterated Tikhonov regularization for bounded operators have been discussed in detail in literature [4] and the reference therein. However, we are dealing with unbounded operators for singularly perturbed problems. Hence, a special treatment is mandatory to address this problem as regularization theory for unbounded operators had received only very little attention in literature except for a few like [7] and reference therein.

To analyze the problem rigorously, we rewrite (1.1) in an operator form as follows

$$Lu = f; \tag{1.3}$$

where $L = \frac{\partial}{\partial t} - \epsilon \frac{\partial^2}{\partial x^2} + a(x,t)$ is a closed unbounded operator acting between suitable Hilbert spaces. The operator equation (1.3) is ill-posed in the sense that, as $\epsilon \to 0$, the solution u is not uniformly convergent to the solution of the reduced problem [(when $\epsilon = 0$ in (1.1)]. In order to motivate the discussion, we will first discuss how iterated Tikhonov regularization can be applied to unbounded operators and then we will analyse the problem in the context of singularly perturbed problems. In order to achieve this goal, let us assume that X is a Hilbert space and $L : X \to X$ be a closed densely defined unbounded linear operator on X satisfying (1.3). The above equation is ill-posed if $N(L) \neq 0$ or $R(L)$ is not closed. Incase only noisy data $\tilde{f}$ with $\|f - \tilde{f}\| < \delta, \delta > 0$ is available instead of f, we consider the operator equation as

$$Lu = \tilde{f}. \tag{1.4}$$

The adjoint of the operator L is defined by, $L^* = -\frac{\partial}{\partial t} - \epsilon \frac{\partial^2}{\partial x^2} + a(x,t)$.

The paper is structured as follows. In Sect. 2, we discuss the general theory for unbounded operators using iterated Tikhonov regularization and in Sect. 3 an a posteriori choice strategy for choosing the regularization parameter is presented. In Sect. 4, we discuss convergence and error estimate analysis for singularly perturbed parabolic problem. Finally, in Sect. 5, we illustrate the efficiency of the approach through numerical examples.

2 Analysis of unbounded operators

In this section, we discuss the general theory of unbounded operators with respect to iterated Tikhonov regularization. Such a discussion is warrant due to the following reasons: (i) Most of the literature on regularization theory is applied to bounded operators; (ii) We need a different treatment to the problem when we deal with data that may not be sufficiently smooth to get a stable solution. Hence, instead of getting a solution using iterated Tikhonov regularizion, as in the case of bounded operators, in the form

$$u_\alpha^n := \sum_{j=1}^{n} \alpha^{j-1} (L^*L + \alpha I)^{-j} L^* f \tag{2.1}$$

for (1.1), we adopt a different approach to get the solution. For unbounded operators, it is not necessary that the data f belong to $D(L^*)$ or that the definition $L^* f$ make any sense at all. This sought us to compute the solution through a different approach and to make an alternate analysis for unbounded operators by considering:

$$u_\alpha^n := \sum_{j=1}^{n} \alpha^{j-1} L^* (LL^* + \alpha I)^{-j} f. \tag{2.2}$$

When we deal with the noisy data $\tilde{f}$, we consider the solution

$$\tilde{u}_\alpha^n := \sum_{j=1}^{n} \alpha^{j-1} L^* (LL^* + \alpha I)^{-j} \tilde{f}. \tag{2.3}$$

Theorem 2.1 *If L is a closed densely defined operator and u_α^n is defined as in (2.2). Then $L^*L(u_\alpha^n - u) \to 0$ as $\alpha \to 0$ and $\|u_\alpha^n\| \le \|u\|$.*

Proof Note that we have $Lu = f$ and $u_\alpha^n := \sum_{j=1}^{n} \alpha^{j-1} L^*(LL^* + \alpha I)^{-j} f$. Therefore, it follow that

$$\begin{aligned} L^*L\left(u_\alpha^n - u\right) &= L^*L\left(\sum_{j=1}^{n} \alpha^{j-1} L^* \left(LL^* + \alpha I\right)^{-j} f - u\right) \\ &= L^*\left(\sum_{j=1}^{n} \alpha^{j-1} LL^* \left(LL^* + \alpha I\right)^{-j} f - Lu\right) \end{aligned}$$

$$
\begin{aligned}
&= L^*\left(\sum_{j=1}^{n} \alpha^{j-1} LL^*\left(LL^*+\alpha I\right)^{-j} f - f\right)\\
&= L^*\left(\sum_{j=1}^{n}\left(\alpha^{j-1}\left(LL^*+\alpha I\right)^{-(j-1)} f - \alpha^{j}\left(LL^*+\alpha I\right)^{-j} f\right) - f\right)\\
&= L^*\left(\left(I-\alpha^{n}\left(LL^*+\alpha I\right)^{-n}\right) f - f\right)\\
&= -\alpha^{n} L^*\left(LL^*+\alpha I\right)^{-n} f.
\end{aligned}
$$

This implies that $\|L^*L(u_\alpha^n - u)\| = \alpha^n\|L^*(LL^*+\alpha I)^{-n} f\|$ and hence $L^*L(u_\alpha^n - u) \to 0$ as $\alpha \to 0$. As above we can rewrite u_α^n as $u_\alpha^n = (I-\alpha^n(L^*L+\alpha I)^{-n})u$ and since $\|I-\alpha^n(L^*L+\alpha I)^{-n}\| \le 1$, we have $\|u_\alpha^n\| \le \|u\|$. This completes the proof. □

Theorem 2.2 *Let u_α^n be defined as in (2.2) and $Lu = f$. Then u_α^n converges to u as $\alpha \to 0$.*

Proof For any $w \in X$ and $\|u_\alpha^n\| \le \|u\|$, we can see that

$$
\begin{aligned}
|\langle u, u_\alpha^n - u\rangle| &\le |\langle L^*Lw, u_\alpha^n - u\rangle| + |\langle u - L^*Lw, u_\alpha^n - u\rangle|\\
&\le |\langle w, L^*L(u_\alpha^n - u)\rangle| + |\langle u - L^*Lw, u_\alpha^n\rangle| + |\langle u - L^*Lw, u\rangle|\\
&\le \|w\|\|L^*L(u_\alpha^n - u)\| + \|u - L^*Lw\|\|u_\alpha^n\| + \|u - L^*Lw\|\|u\|\\
&\le \|w\|\|L^*L(u_\alpha^n - u)\| + 2\|u - L^*Lw\|\|u\|.
\end{aligned}
$$

Using Theorem 2.1 and the fact that $\overline{R(L^*L)} = N(L)^\perp$, we see that $\langle u, u_\alpha^n - u\rangle \to 0$ as $\alpha \to 0$ and hence $\langle u, u_\alpha^n\rangle \to \|u\|^2$ as $\alpha \to 0$. Again by using Theorem 2.1, we obtain that

$$
\begin{aligned}
\|u_\alpha - u\|^2 &= \|u_\alpha\|^2 - 2\langle u_\alpha, u\rangle + \|u\|^2\\
&\le 2\|u\|^2 - 2\langle u_\alpha, u\rangle \to 0, \text{ as } \alpha \to 0.
\end{aligned}
$$

Therefore, $\|u_\alpha^n - u\| \to 0$ as $\alpha \to 0$. □

Theorem 2.3 *(A priori Method) Let u_α^n and $\tilde{u}_\alpha^n$ be defined according to (2.2) and (2.3) respectively. If $\tilde{f}$ is such that $\|f - \tilde{f}\| \le \delta$, then $\|\tilde{u}_\alpha^n - u\| \le q(\alpha) + c\frac{\delta}{\sqrt{\alpha}}$, where $q(\alpha) = \|\alpha^n(L^*L+\alpha I)^{-n}u\|$. Furthermore, if the regularization parameter α is chosen with condition that $\alpha(\delta) \to 0$ and $\frac{\delta}{\sqrt{\alpha(\delta)}} \to 0$, then $\|\tilde{u}_\alpha^n - u\| \to 0$ as $\delta \to 0$. If $u \in R((L^*L)^\nu)$, $0 < \nu \le n$, then*

$$\|\tilde{u}_\alpha^n - u\| \le \alpha^\nu + c\frac{\delta}{\sqrt{\alpha}}$$

and by choosing $\alpha \sim \delta^{2/(2\nu+1)}$

$$\|\tilde{u}_\alpha^n - u\| = O(\delta^{2\nu/(2\nu+1)}).$$

Proof First note that, $\tilde{u}_\alpha^n - u = u_\alpha^n - u + \tilde{u}_\alpha^n - u_\alpha^n$ which implies

$$
\begin{aligned}
\|\tilde{u}_\alpha^n - u\| &\leq \|u_\alpha^n - u\| + \|\tilde{u}^n - u_\alpha^n\| \\
&= \left\| \sum_{j=1}^{n} \alpha^{j-1} L^*(LL^* + \alpha I)^{-j} f - u \right\| \\
&\quad + \left\| \sum_{j=1}^{n} \alpha^{j-1} L^*(LL^* + \alpha I)^{-j} f - \sum_{j=1}^{n} \alpha^{j-1} L^*(LL^* + \alpha I)^{-j} \tilde{f} \right\| \\
&= \left\| \sum_{j=1}^{n} \alpha^{j-1} L^*(LL^* + \alpha I)^{-j} Lu - u \right\| \\
&\quad + \left\| \sum_{j=1}^{n} \alpha^{j-1} L^*(LL^* + \alpha I)^{-j} (f - \tilde{f}) \right\| \\
&\leq \left\| \alpha^n (L^*L + \alpha I)^{-n} u \right\| + \left\| \sum_{j=1}^{n} \alpha^{j-1} L^*(LL^* + \alpha I)^{-j} \right\| \left\| f - \tilde{f} \right\| \\
&\leq q(\alpha) + c\frac{\delta}{\sqrt{\alpha}}. \qquad (2.4)
\end{aligned}
$$

In Theorem 2.2, we have seen that $q(\alpha) \to 0$ as $\alpha \to 0$. For $\delta > 0$, choose $\alpha = \alpha(\delta)$ such that $\alpha(\delta) \to 0$ and $\frac{\delta}{\sqrt{\alpha(\delta)}} \to 0$. Therefore, if $\delta \to 0$ implies that $q(\alpha) + c\frac{\delta}{\sqrt{\alpha}} \to 0$.

Using spectral theory result, one can easily prove that $q(\alpha) \leq \alpha^\nu$ and hence the result. □

3 An a posteriori parameter choice rule

In this section, we investigate an a posteriori parameter choice rule to get an optimal rate of convergence for the solution of ill-posed unbounded operator equation. The parameter choice rule for bounded operators is discussed in detail in [4,5] and reference therein. The parameter α is chosen in such a way that it depends on both data and noise level δ and, we consider the following discrepancy principle for choosing the regularization parameter.

$$g(\alpha, \tilde{f}) := 2\alpha^{2n+1}\langle (LL^* + \alpha I)^{-(2n+1)}\tilde{f}, \tilde{f}\rangle = \gamma^2\delta^2, \qquad (3.1)$$

$$\text{where } 0 < \gamma^2\delta^2 \leq 2\|\tilde{f}\|^2. \qquad (3.2)$$

First we claim that there exists a unique α satisfying the discrepancy principle.

Theorem 3.1 *Assume that $\gamma > \sqrt{2}$, $\tilde{f} \neq 0$ and $0 < \gamma\delta \leq 2\|\tilde{f}\|$. Then there exists a unique α satisfies $g(\alpha, \tilde{f}) = \gamma^2\delta^2$. Furthermore, $g(\alpha, \tilde{f})$ is continuously differentiable and strictly increasing and its derivative is $g'(\alpha, \tilde{f}) = 2(2n+1)\alpha^{2n}\|L^*(LL^* + \alpha I)^{-(n+1)}\tilde{f}\|^2$.*

Proof Due to spectral representation theorem, we obtain that

$$
\begin{aligned}
g(\alpha, \tilde{f}) &= 2\alpha^{2n+1}\langle (LL^* + \alpha I)^{-(2n+1)}\tilde{f}, \tilde{f}\rangle \\
&= 2\int_0^\infty \left(\frac{\alpha}{\lambda+\alpha}\right)^{2n+1} d\langle F_\lambda \tilde{f}, F_\lambda \tilde{f}\rangle.
\end{aligned}
$$

Using the Dominated convergence theorem and the fact $\frac{\alpha}{\lambda+\alpha} < 1$, we deduce the below relations

$$
\begin{aligned}
\lim_{\alpha\to\infty} g(\alpha, \tilde{f}) &= \lim_{\alpha\to\infty} 2\int_0^\infty \left(\frac{\alpha}{\lambda+\alpha}\right)^{2n+1} d\langle F_\lambda \tilde{f}, F_\lambda \tilde{f}\rangle \\
&= 2\int_0^\infty \lim_{\alpha\to\infty}\left(\frac{\alpha}{\lambda+\alpha}\right)^{2n+1} d\langle F_\lambda \tilde{f}, F_\lambda \tilde{f}\rangle \\
&= 2\|\tilde{f}\|^2.
\end{aligned}
$$

As above, we have

$$
\begin{aligned}
\lim_{\alpha\to 0} g(\alpha, \tilde{f}) &= 2\int_0^\infty \lim_{\alpha\to 0}\left(\frac{\alpha}{\lambda+\alpha}\right)^{2n+1} d\langle F_\lambda \tilde{f}, F_\lambda \tilde{f}\rangle \\
&= 2\|P_{N(LL^*)}\tilde{f}\|^2,
\end{aligned}
$$

where $\{F_\lambda\}$ and $P_{N(LL^*)}$ are spectral family of an operator LL^*, an orthogonal projection on $N(LL^*)$ respectively. If $f \in R(L)$ satisfying $\|f - \tilde{f}\| \leq \delta$, then

$$
\|P_{N(LL^*)}\tilde{f}\| \leq \|P_{N(LL^*)}f\| + \|P_{(NLL^*)}(f - \tilde{f})\| \leq \delta
$$

and accordingly $\lim_{\alpha\to 0} g(\alpha, \tilde{f}) \leq 2\delta^2$. Therefore,

$$
\lim_{\alpha\to 0} \alpha^{2n+1}\int_0^\infty \left(\frac{1}{\lambda+\alpha(\delta)}\right)^{2n+1} d\langle F_\lambda \tilde{f}, \tilde{f}\rangle = \lim_{\alpha\to 0} g(\alpha, \tilde{f}) = 0
$$

as $\delta \to 0$. We can conclude from above that $\lim_{\delta\to 0}\alpha(\delta) = 0$. It can be seen that $g(\alpha, \tilde{f})$ is continuously differentiable and the derivative of $g(\alpha, \tilde{f})$,

$$
g'(\alpha, \tilde{f}) = 2(2n+1)\alpha^{2n}\|L^*(LL^* + \alpha I)^{-(n+1)}\tilde{f}\|^2
$$

and hence $g(\alpha, \tilde{f})$ is strictly increasing. Now since $0 < \gamma\delta \leq 2\|\tilde{f}\|$, by intermediate value theorem, there exists a unique α such that $g(\alpha(\delta), \tilde{f}) = \gamma^2\delta^2$. □

Next we discuss the parameter choice rule with the exact data.

Theorem 3.2 *Suppose that $\gamma > \sqrt{2}$, $\gamma_1 = \gamma - \sqrt{2}$, $\gamma_2 = \gamma + \sqrt{2}$ and $\hat{\alpha}$ is computed through (3.1), then there exist a constant $\gamma \in [\gamma_1, \gamma_2]$ such that $g(\hat{\alpha}, f) = \gamma^2\delta^2$.*

Proof From (3.1), we have

$$\begin{aligned} g^{1/2}(\alpha, f) &= \sqrt{2}\alpha^{(2n+1)/2} \left\| (LL^* + \alpha I)^{-(2n+1)/2} f \right\| \\ &= \sqrt{2}\alpha^{(2n+1)/2} \left\| (LL^* + \alpha I)^{-(2n+1)/2} \tilde{f} \right. \\ &\quad \left. + (LL^* + \alpha I)^{-(2n+1)/2}(f - \tilde{f}) \right\| \\ &\lesseqgtr \sqrt{2}\alpha^{(2n+1)/2} \left\| (LL^* + \alpha I)^{-(2n+1)/2} \tilde{f} \right\| \\ &\quad \pm \sqrt{2}\alpha^{(2n+1)/2} \left\| (LL^* + \alpha I)^{-(2n+1)/2}(f - \tilde{f}) \right\|. \end{aligned}$$

Since $\|f - \tilde{f}\| \leq \delta$, we have that $(\gamma - \sqrt{2})^2\delta^2 \leq g(\alpha, f) \leq (\gamma + \sqrt{2})^2\delta^2$. Thus, there exist a constant $\gamma \in [\gamma_1, \gamma_2]$ satisfying $g(\hat{\alpha}, f) = \gamma^2\delta^2$. □

Theorem 3.3 *Let $\gamma \in [\gamma_1, \gamma_2]$ with $\gamma > \sqrt{2}$, $\gamma_1 = \gamma - \sqrt{2}$ and $\gamma_2 = \gamma + \sqrt{2}$, then $p(\alpha) = q(\alpha)^2 + \gamma^2\frac{\delta^2}{\alpha}$ attains minimum if and only if $g(\alpha, f) = \gamma^2\delta^2$.*

Proof Proof follows from properties of $p'(\alpha)$ and Theorem 3.2. □

Now we see that α obtained from (3.1) gives the minimum error.

Theorem 3.4 *Suppose $\gamma \in [\gamma_1, \gamma_2]$ with $\gamma > \sqrt{2}$, $\gamma_1 = \gamma - \sqrt{2}$, $\gamma_2 = \gamma + \sqrt{2}$ and $\hat{\alpha}$ is computed through (3.1), then*

$$\|\tilde{u}^n_{\hat{\alpha}} - u\| \leq c_0 \inf_{\alpha > 0} \left\{ q(\alpha) + c\frac{\delta}{\sqrt{\alpha}} \right\} \tag{3.3}$$

where $c_0^2 = 2\max\{(c/\gamma_1)^2, (\gamma_2/c)^2\}$.

Proof Using (2.4), Theorem 3.3 and $\hat{\alpha}$ obtained from (3.1), we see that $\|\tilde{u}^n_{\hat{\alpha}} - u\| \leq q(\hat{\alpha}) + c\frac{\delta}{\sqrt{\hat{\alpha}}}$. Squaring both sides yields,

$$\begin{aligned} \|\tilde{u}^n_{\hat{\alpha}} - u\|^2 &\leq 2\left(q(\hat{\alpha})^2 + c^2\frac{\delta^2}{\hat{\alpha}} \right) \\ &\leq 2\max\{(c/\gamma)^2, 1\} \left(q(\hat{\alpha})^2 + \gamma^2\frac{\delta^2}{\hat{\alpha}} \right) \\ &\leq 2\max\{(c/\gamma)^2, 1\} \inf_{\alpha>0} \left\{ q(\alpha)^2 + \gamma^2\frac{\delta^2}{\alpha} \right\} \\ &\leq 2\max\{(c/\gamma)^2, 1\} \max\{(\gamma/c)^2, 1\} \inf_{\alpha>0} \left\{ q(\alpha)^2 + c^2\frac{\delta^2}{\alpha} \right\} \\ &\leq 2\max\{(c/\gamma_1)^2, (\gamma_2/c)^2\} \inf_{\alpha>0} \left\{ q(\alpha)^2 + c^2\frac{\delta^2}{\alpha} \right\}. \end{aligned}$$

Thus, $\|\tilde{u}^n_{\hat{\alpha}} - u\| \leq c_0 \inf_{\alpha>0} \left\{q(\alpha) + c\frac{\delta}{\sqrt{\alpha}}\right\}$ with $c_0^2 = 2\max\{(c/\gamma_1)^2, (\gamma_2/c)^2\}$. □

Theorem 3.5 *Assume that* $u \in R((L^*L)^\nu)$ *with* $0 < \nu \leq n$. *Then*

1. *There exists a unique* α *such that* $\alpha^{(2n+1)/2}\|(L^*L + \alpha I)^{-n}(L^*L)^\nu w\| = c\delta$.
2. $\|\tilde{u}^n_{\hat{\alpha}} - u\| = O\left(\delta^{\frac{2\nu}{(2\nu+1)}}\right)$.

Proof By Theorem 3.4, we can see that

$$\|\tilde{u}^n_{\hat{\alpha}} - u\| \leq c_0 \inf_{\alpha>0} \left\{\alpha^n\|(L^*L + \alpha I)^{-n}u\| + c\frac{\delta}{\sqrt{\alpha}}\right\}.$$

If $u \in R((L^*L)^\nu)$, then

$$\|\tilde{u}^n_{\hat{\alpha}} - u\| \leq c_0 \inf_{\alpha>0} \left\{\alpha^n\|(L^*L + \alpha I)^{-n}(L^*L)^\nu w\| + c\frac{\delta}{\sqrt{\alpha}}\right\}.$$

Consider $g_1(\alpha) = \alpha^{(2n+1)/2}\|(L^*L+\alpha I)^{-n}(L^*L)^\nu w\|$. In the similar fashion (as in the proof of the Theorem 3.1), we obtain that $\lim_{\alpha\to 0} g_1(\alpha) = 0$, $\lim_{\alpha\to\infty} g_1(\alpha) = \infty$, $g_1(\alpha)$ is continuous and increasing and therefore, there exist a unique α such that $g_1(\alpha) = c\delta$. Thus,

$$\begin{aligned}
\|\tilde{u}^n_{\hat{\alpha}} - u\| &= O\left(\frac{\delta}{\sqrt{\alpha}}\right) \\
&= O\left(\delta^{\frac{2\nu}{2\nu+1}}\delta^{\frac{1}{2\nu+1}}\frac{1}{\sqrt{\alpha}}\right) \\
&= O\left(\delta^{\frac{2\nu}{2\nu+1}}\left(\frac{\delta}{\sqrt{\alpha}}\right)^{\frac{1}{2\nu+1}}\left(\frac{1}{\sqrt{\alpha}}\right)^{\frac{2\nu}{2\nu+1}}\right) \\
&= O\left(\delta^{\frac{2\nu}{2\nu+1}}\left\|\alpha^n(L^*L + \alpha I)^{-n}(L^*L)^\nu v\right\|^{\frac{1}{2\nu+1}}\left(\frac{1}{\alpha}\right)^{\frac{\nu}{2\nu+1}}\right) \\
&= O\left(\delta^{\frac{2\nu}{2\nu+1}}\left\|\alpha^{n-\nu}(L^*L + \alpha I)^{-n}(L^*L)^\nu v\right\|^{\frac{1}{2\nu+1}}\right) \\
&= O\left(\delta^{\frac{2\nu}{2\nu+1}}\right).
\end{aligned}$$

In particular, if $\nu = n$, then $\|\tilde{u}^n_{\hat{\alpha}} - u\| = O\left(\delta^{\frac{2n}{2n+1}}\right)$. □

4 Application to the singularly perturbed 1D parabolic PDE

In this section, we apply the theory developed in previous section to singularly perturbed 1D parabolic reaction-diffusion problem:

$$\frac{\partial u}{\partial t} - \epsilon \frac{\partial^2 u}{\partial x^2} + a(x,t)u = f(x,t), \;\; \Omega := (0,1) \times (0,1), \tag{4.1}$$

$$u(x,0) = 0, \;\; u(0,t) = u(1,t) = 0, \;\; t, x \in [0,1] \tag{4.2}$$

where $0 < \epsilon << 1$ and the reaction term satisfies $0 < \beta \le a(x,t)$ for all $(x,t) \in \bar{\Omega}$ and the data f are sufficiently smooth functions on $\bar{\Omega}$. This we represented as

$$Lu = f. \tag{4.3}$$

Theorem 4.1 *Let $u_\alpha^n = \sum_{j=1}^n \alpha^{j-1} L^*(LL^* + \alpha I)^{-1} f$ and u be the iterative regularized approximate solution and the solution of the singularly perturbed parabolic PDE (1.3) respectively. Then u_α converges to u as $\alpha \to 0$.*

Proof Proof is analogous to the proof of the Theorem 2.2. □

Theorem 4.2 *Let u be the solution of the singularly perturbed parabolic PDE (1.3) and $u_\alpha^n = \sum_{j=1}^n \alpha^{j-1} L^*(LL^* + \alpha I)^{-j} f$ and $\tilde{u}_\alpha^n = \sum_{j=1}^n \alpha^{j-1} L^*(LL^* + \alpha I)^{-j} \tilde{f}$ be the regularized solution with respect to the actual data f and perturbed data $\tilde{f}$ respectively. Then the following hold good.*

1. *$\|\tilde{u}_\alpha^n - u\| \le q(\alpha) + c\frac{\delta+\epsilon}{\sqrt{\alpha}}$, where $q(\alpha) = \|\alpha^n (L^*L + \alpha I)^{-n} u\|$.*
2. *(A priori Method) If the regularization parameter α that depends on both δ, ϵ satisfying $\alpha(\delta,\epsilon) \to 0$ and $\frac{\delta+\epsilon}{\sqrt{\alpha(\delta)}} \to 0$, then $\|\tilde{u}_\alpha^n - u\| \to 0$ as δ, $\epsilon \to 0$. Moreover, if $u \in R((L^*L)^\nu)$, $0 < \nu \le n$, then*

$$\|\tilde{u}_\alpha^n - u\| \le \alpha^\nu + c\frac{\delta + \epsilon}{\sqrt{\alpha}}$$

and by choosing $\alpha \sim (\delta + \epsilon)^{2/(2\nu+1)}$

$$\|\tilde{u}_\alpha^n - u\| = O((\delta + \epsilon)^{2\nu/(2\nu+1)}).$$

Proof It is easy to see that,

$$\begin{aligned}
\left\|\tilde{u}_\alpha^n - u\right\| &\le \left\|u_\alpha^n - u\right\| + \left\|\tilde{u}_\alpha^n - u_\alpha^n\right\| \\
&= \left\|\sum_{j=1}^n \alpha^{j-1} L^*(LL^* + \alpha I)^{-j} Lu - u\right\| \\
&\quad + \left\|\sum_{j=1}^n \alpha^{j-1} L^*(LL^* + \alpha I)^{-j} (f - \tilde{f})\right\| \\
&\le \left\|\alpha^n (L^*L + \alpha I)^{-n} u\right\| + \left\|\sum_{j=1}^n \alpha^{j-1} L^*(LL^* + \alpha I)^{-j}\right\| \left\|f - \tilde{f}\right\| \\
&\le q(\alpha) + c\frac{\delta}{\sqrt{\alpha}}
\end{aligned}$$

$$\leq q(\alpha) + c\frac{\delta + \epsilon}{\sqrt{\alpha}}, \tag{4.4}$$

where $q(\alpha) = \|\alpha^n(L^*L + \alpha I)^{-n}u\|$, which proves the first assertion. It is known from Theorem 4.1, $q(\alpha)$ tends to zero as $\alpha \to 0$. For a given $\delta > 0$, $\epsilon > 0$, choose $\alpha = \alpha(\delta, \epsilon)$ fulfilling $\alpha(\delta, \epsilon) \to 0$ and $\frac{\delta+\epsilon}{\sqrt{\alpha(\delta)}} \to 0$. Thus, $q(\alpha) + c\frac{\delta+\epsilon}{\sqrt{\alpha}} \to 0$ as $\delta, \epsilon \to 0$. The final result follows from the spectral theory and hence the proof of the theorem. □

In order to choose the regularization parameter for singularly perturbed parabolic PDE in an a posteriori manner, we slightly modify the parameter choice rule (3.1) by taking into account the singularly perturbed parameter. The parameter choice rule for singularly perturbed parabolic PDE is considered as:

$$g(\alpha, \tilde{f}) := 2\alpha^{2n+1}\langle (LL^* + \alpha I)^{-(2n+1)}\tilde{f}, \tilde{f}\rangle = \gamma^2(\delta + \epsilon)^2 \tag{4.5}$$

$$\text{where } 0 < \gamma^2(\delta + \epsilon)^2 \leq 2\|\tilde{f}\|^2. \tag{4.6}$$

The following theorems emphasis that we can find out a unique α depends on the singularly perturbed parameter ϵ and the noise level δ.

Theorem 4.3 *Suppose that $\gamma > \sqrt{2}$ and $\tilde{f} \neq 0$. The function defined in (4.5) satisfies the following properties.*

1. *Continuously differentiable, strictly increasing and $g'(\alpha, \tilde{f}) = 2(2n+1)\alpha^{2n}\|L^*(LL^* + \alpha I)^{-(n+1)}\tilde{f}\|^2$.*
2. *If $0 < \gamma^2(\delta+\epsilon)^2 \leq 2\|\tilde{f}\|^2$, then there exists a unique α satisfying $g(\alpha(\delta, \epsilon), \tilde{f}) = \gamma^2(\delta + \epsilon)^2$.*

Proof Proof is analogous to the proof of the Theorem 3.1. □

Theorem 4.4 *Let γ, $\gamma_1$and γ_2 be defined in Theorem 3.2. If $\hat{\alpha}$ is determined by (4.5), then there exist a constant $\gamma \in [\gamma_1, \gamma_2]$ such that $g(\hat{\alpha}, f) = \gamma^2(\delta + \epsilon)^2$.*

Proof By the definition of g, it is easy to see that

$$\begin{aligned} g^{1/2}(\alpha, f) &= \sqrt{2}\alpha^{(2n+1)/2}\left\|(LL^* + \alpha I)^{-(2n+1)/2}f\right\| \\ &= \sqrt{2}\alpha^{(2n+1)/2}\left\|(LL^* + \alpha I)^{-(2n+1)/2}\tilde{f}\right. \\ &\qquad \left. + (LL^* + \alpha I)^{-(2n+1)/2}(f - \tilde{f})\right\| \\ &\lesseqgtr \sqrt{2}\alpha^{(2n+1)/2}\left\|(LL^* + \alpha I)^{-(2n+1)/2}\tilde{f}\right\| \\ &\qquad \pm \sqrt{2}\alpha^{(2n+1)/2}\left\|(LL^* + \alpha I)^{-(2n+1)/2}(f - \tilde{f})\right\|. \end{aligned}$$

If $\|f - \tilde{f}\| \leq \delta$, then $(\gamma - \sqrt{2})^2(\delta + \epsilon)^2 \leq g(\alpha, f) \leq (\gamma + \sqrt{2})^2(\delta + \epsilon)^2$. By Theorem 4.3, there exist a constant $\gamma \in [\gamma_1, \gamma_2]$ satisfying $g(\hat{\alpha}, f) = \gamma^2(\delta + \epsilon)^2$. □

Theorem 4.5 *Let γ, $\gamma_1$and γ_2 be defined in Theorem 3.2. Then $p(\alpha) = q(\alpha)^2 + \gamma^2\frac{(\delta+\epsilon)^2}{\alpha}$ attains minimum if and only if $g(\alpha, f) = \gamma^2(\delta+\epsilon)^2$.*

Proof Proof follows from the Theorem 4.4 and the properties of $p'(\alpha)$. □

Now we show that α obtained through discrepancy principle (4.5) gives the minimum error.

Theorem 4.6 *Let γ, γ_1, and γ_2 be given in Theorem 3.4. The parameter $\hat{\alpha}$ is chosen according to (4.5), then*

$$\|\tilde{u}^n_{\hat{\alpha}} - u\| \le c_0 \inf_{\alpha>0}\left\{q(\alpha) + c\frac{\delta+\epsilon}{\sqrt{\alpha}}\right\} \tag{4.7}$$

where $c_0^2 = 2\max\{(c/\gamma_1)^2, (\gamma_2/c)^2\}$.

Proof From (4.4) we have $\|\tilde{u}^n_{\hat{\alpha}} - u\| \le q(\hat{\alpha}) + c\frac{\delta}{\sqrt{\hat{\alpha}}}$. Since α obtained through discrepancy principle (4.5) gives the minimum error, by squaring both sides of the above relation we have,

$$\begin{aligned}
\|\tilde{u}^n_{\hat{\alpha}} - u\|^2 &\le 2\left(q(\hat{\alpha})^2 + c^2\frac{(\delta+\epsilon)^2}{\hat{\alpha}}\right)\\
&\le 2\max\{(c/\gamma)^2, 1\}\left(q(\hat{\alpha})^2 + \gamma^2\frac{(\delta+\epsilon)^2}{\hat{\alpha}}\right)\\
&\le 2\max\{(c/\gamma)^2, 1\}\inf_{\alpha>0}\left\{q(\alpha)^2 + \gamma^2\frac{(\delta+\epsilon)^2}{\alpha}\right\}\\
&\le 2\max\{(c/\gamma)^2, 1\}\max\{(\gamma/c)^2, 1\}\inf_{\alpha>0}\left\{q(\alpha)^2 + c^2\frac{(\delta+\epsilon)^2}{\alpha}\right\}\\
&\le 2\max\{(c/\gamma_1)^2, (\gamma_2/c)^2\}\inf_{\alpha>0}\left\{q(\alpha)^2 + c^2\frac{(\delta+\epsilon)^2}{\alpha}\right\}.
\end{aligned}$$

Hence, $\|\tilde{u}^n_{\hat{\alpha}} - u\| \le c_0 \inf_{\alpha>0}\left\{q(\alpha) + c\frac{\delta+\epsilon}{\sqrt{\alpha}}\right\}$ with $c_0^2 = 2\max\{(c/\gamma_1)^2, (\gamma_2/c)^2\}$. □

Theorem 4.7 *Suppose that $u \in R((L^*L)^\nu)$ with $0 < \nu \le n$. Then*

1. *There exists a unique α such that $\alpha^{(2n+1)/2}\|(L^*L+\alpha I)^{-n}(L^*L)^\nu w\| = c(\delta+\epsilon)$.*
2. $\|\tilde{u}^n_{\hat{\alpha}} - u\| = O\left((\delta+\epsilon)^{\frac{2\nu}{2\nu+1}}\right)$.

Proof Proof is analogous to the proof of the Theorem 3.5. □

Theorem 4.8 *Let L be a singular perturbed parabolic operator defined in (1.3) and $u^n_\alpha = \sum_{j=1}^n \alpha^{j-1}L^*(LL^*+\alpha I)^{-j} f$ be the iterative regularized approximate solution of (1.3). Then, the following result holds for a large positive c,*

$$|u^n_\alpha(x,t)| \le c\left(1 + e^{-\sqrt{\frac{\beta}{\epsilon}}x} + e^{-\sqrt{\frac{\beta}{\epsilon}}(1-x)}\right).$$

Table 1 Sup norm and L^2-norm errors for $a = 1$ with out data error

ϵ	Norm and α	$n = 1$	$n = 2$	$n = 3$	$n = 4$
10^{-2}	α	6.3687e−02	3.9146e−01	1.3397e−01	3.8762e−01
	e_α(Sup.)	2.6377e−03	2.7652e−03	2.7696e−03	2.7696e−03
	e_α (L^2)	2.2426e−02	2.7777e−02	2.8357e−02	2.8357e−02
10^{-4}	α	2.1946e−02	2.9220e−01	2.3131e−01	4.4231e−01
	e_α(Sup.)	1.1621e−02	2.0405e−03	2.3053e−03	2.3107e−03
	e_α (L^2)	1.1624e−01	1.1781e−02	1.3187e−02	1.3228e−02
10^{-6}	α	2.4072e−02	1.2275e−01	2.0569e−01	6, 1188e−01
	e_α(Sup.)	1.3471e−03	4.5171e−05	5.9304e−05	5.9564e−05
	e_α (L^2)	1.8712e−03	3.1587e−04	3.4027e−04	3.4182e−04
10^{-8}	α	4.1551e−03	2.8170e−02	4.8407e−01	9.0246e−01
	e_α(Sup.)	2.3935e−04	6.0213e−07	5.7176e−07	5.5312e−07
	e_α (L^2)	3.2586e−03	7.5664e−06	3.2908e−06	3.1945e−06
10^{-10}	α	4.1961e−03	4.5122e−02	4.8453e−01	9.0431e−01
	e_α(Sup.)	2.4224e−04	9.9874e−07	4.1729e−08	3.1908e−09
	e_α (L^2)	3.2921e−03	1.3482e−05	5.6239e−07	4.2888e−08

Table 2 Sup norm and L^2-norm errors for $a = 1 + xe^{-t}$ with out data error

ϵ	Norm and α	$n = 1$	$n = 2$	$n = 3$	$n = 4$
10^{-2}	α	5.8124e−02	2.1729e−01	1.2316e−01	6.2239e−01
	e_α(Sup.)	2.6489e−03	2.7631e−03	2.7652e−03	2.7652e−03
	e_α (L^2)	2.1351e−02	2.6081e−02	2.6316e−02	2.6318e−02
10^{-4}	α	2.0678e−01	3.0206e−01	4.7687e−01	5.0320e−01
	e_α(Sup.)	1.0836e−02	1.9950e−03	2.2446e−03	2.2551e−03
	e_α (L^2)	1.4130e−01	8.6931e−03	9.6688e−0	9.7316e−03
10^{-6}	α	2.3583e−02	1.2670e−01	3.7330e−01	5.1670e−01
	e_α(Sup.)	1.3068e−03	4.3946e−05	5.7817e−05	5.8284e−05
	e_α (L^2)	1.6913e−02	2.4512e−04	2.5349e−04	2.5570e−04
10^{-8}	α	2.7615e−03	2.9651e−02	1.7935e−01	8.3919e−01
	e_α(Sup.)	1.5814e−04	4.1194e−07	5.7770e−07	5.8386e−07
	e_α (L^2)	1.9943e−03	4.5738e−06	2.5336e−06	2.5625e−06
10^{-10}	α	2.8688e−03	4.8183e−02	2.3833e−01	8.4479e−01
	e_α(Sup.)	1.6481e−04	7.2006e−07	1.4828e−08	4.7774e−09
	e_α (L^2)	2.0728e−03	3.3845e−06	1.5944e−07	2.3121e−08

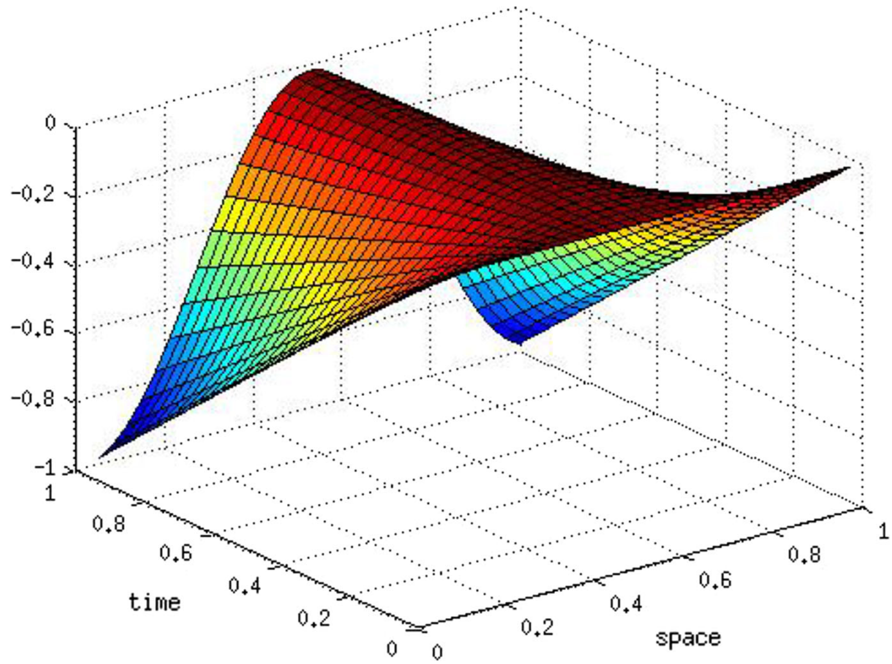

Fig. 1 Actual solution for $\epsilon = 10^{-8}$

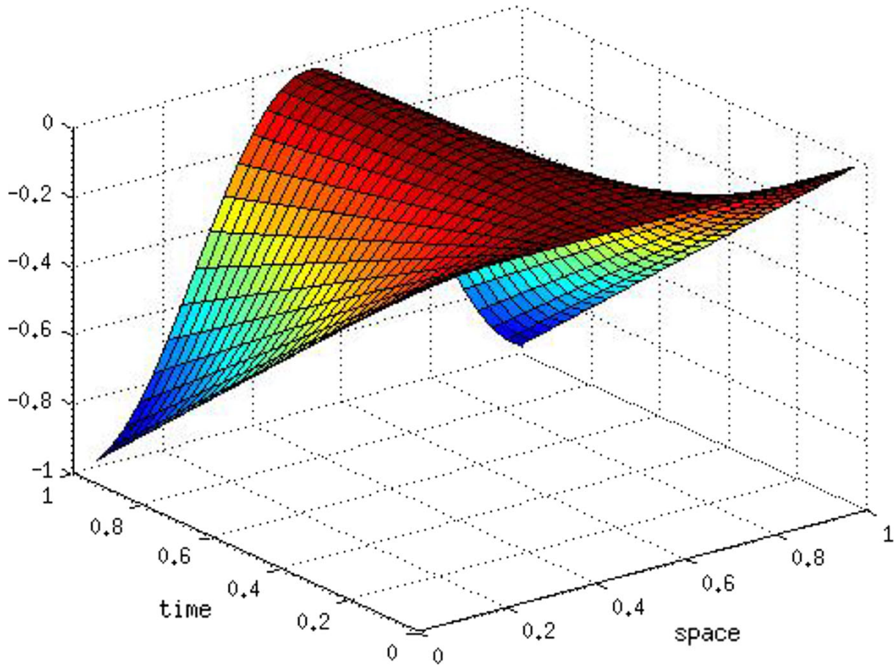

Fig. 2 Computed solution for $\epsilon = 10^{-8}$

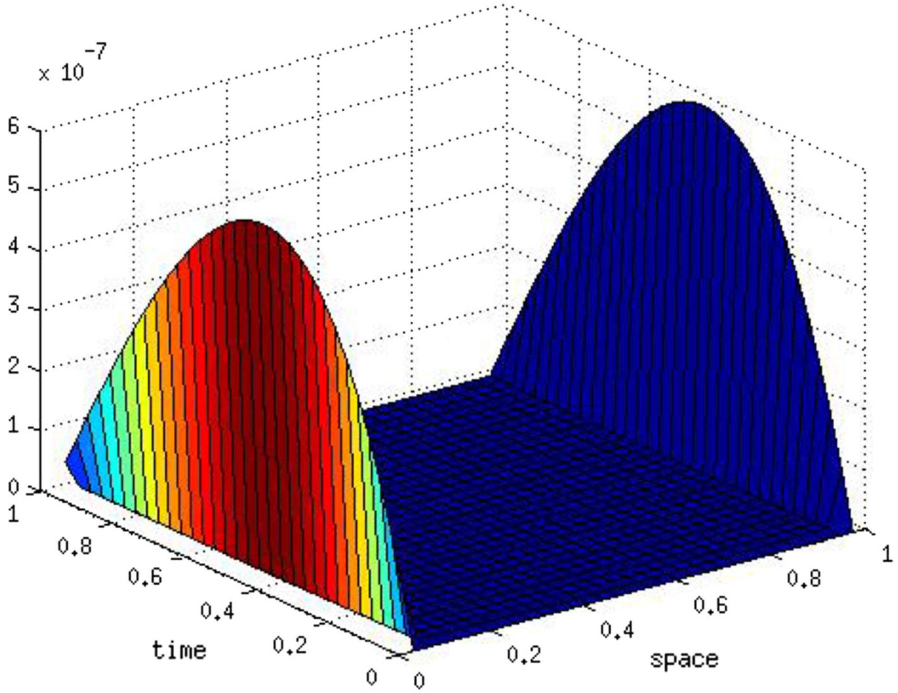

Fig. 3 Reg error for $a = 1$ and $\epsilon = 10^{-8}$

Proof Let $u_\alpha^n = \sum_{j=1}^{n} \alpha^{j-1} L^*(LL^* + \alpha I)^{-j} f$, then we can rewrite it as $u_\alpha^n = (I - \alpha^n (L^*L + \alpha I)^{-n})u$. Let $g_2(x) = c(1 + e^{-\sqrt{\frac{\beta}{\epsilon}}x} + e^{-\sqrt{\frac{\beta}{\epsilon}}(1-x)})$, we see that

$$\begin{aligned} L(g_2 \pm u_\alpha) &= Lg \pm Lu_\alpha^n \\ &= Lg_2 \pm L(I - \alpha^n (L^*L + \alpha I)^{-n})u \\ &\geq \beta c \pm (f - \alpha^n (LL^* + \alpha I)^{-n}) f. \end{aligned}$$

For a large positive c, we obtain that $L(g_2 \pm u_\alpha) \geq 0$ on Ω. Using weak maximum principle, we have $g_2 \pm u_\alpha^n \geq 0$ on $\overline{\Omega}$. □

Theorem 4.9 *Let u be the solution of singular perturbed parabolic operator defined in (1.3) and $u_\alpha^n = \sum_{j=1}^{n} \alpha^{j-1} L^*(LL^* + \alpha I)^{-j} f$ be the iterative regularized approximate solution of (1.3). Then, it holds for a large positive c.*

$$|u_\alpha^n(x,t) - u| \leq c\alpha \left(1 + e^{-\sqrt{\frac{\beta}{\epsilon}}x} + e^{-\sqrt{\frac{\beta}{\epsilon}}(1-x)}\right).$$

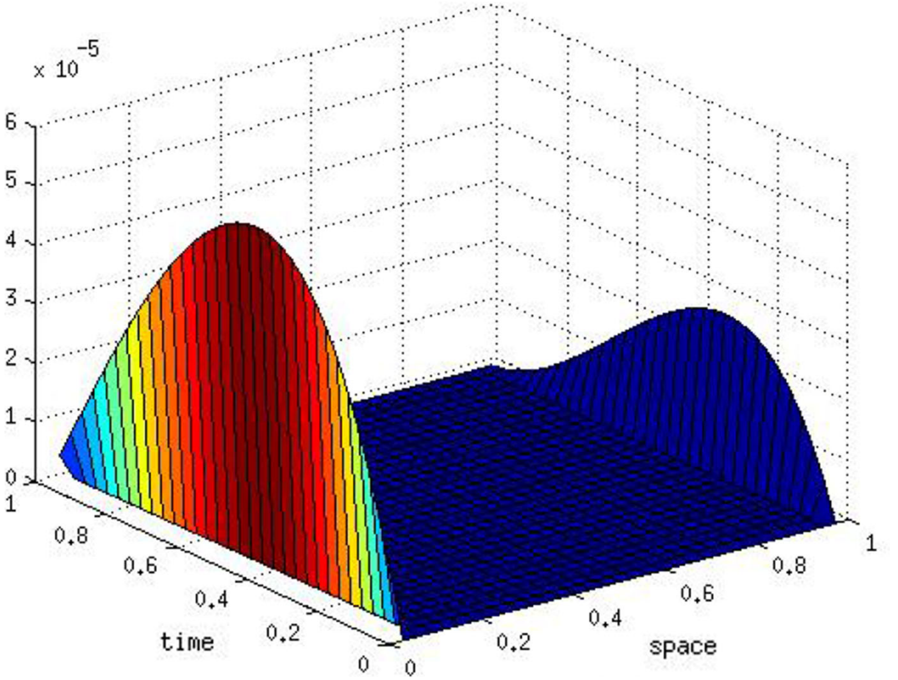

Fig. 4 Reg error for $a = 1 + xe^{-t}$ and $\epsilon = 10^{-6}$

Proof From the proof of the Theorem 4.8, $u_\alpha^n = (I - \alpha^n(L^*L + \alpha I)^{-n})u$. Using $g_2(x) = c\alpha(1 + e^{-\sqrt{\frac{\beta}{\epsilon}}x} + e^{-\sqrt{\frac{\beta}{\epsilon}}(1-x)})$, it is easy to see that

$$\begin{aligned} L(g_2 \pm (u - u_\alpha^n)) &= Lg_2 \pm L(u - u_\alpha^n) \\ &= -\beta(g_2 - c\alpha) + a(x,t)g_2 \pm \alpha^n L(L^*L + \alpha)^{-n}u \\ &\geq -\beta(g_2 - c\alpha) + \beta g_2 \pm \alpha^n (LL^* + \alpha)^{-n} f \\ &= c\alpha\beta \pm \alpha^n (LL^* + \alpha)^{-n} f. \end{aligned}$$

We have that $L(g_2 \pm (u - u_\alpha^n)) \geq 0$ on Ω, for a large $c > 0$. From weak maximum principle, $g_2 \pm (u - u_\alpha^n) \geq 0$ on $\overline{\Omega}$. Hence, $|u - u_\alpha^n| \leq g_2$. □

5 Numerical examples

This section deals with numerical implementation of the regularized equation to examine the theoretical results and its applicability. The numerical computations are done in MATLAB. The regularization parameter α is chosen through both a priori and a posteriori parameter choice rule (4.5). We consider two different cases for discussion, one with no error in the data f and another with a perturbed data $\tilde{f}$ of f by introducing the random error. The respective parameter choice rule applicable in these cases will

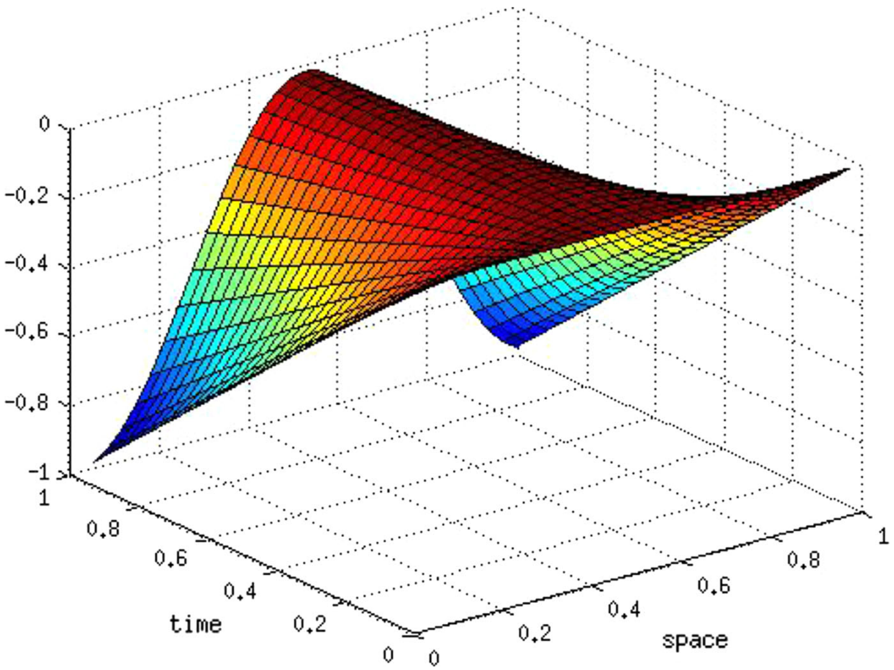

Fig. 5 Act sol for $a = 1 + xe^{-t}$ and $\epsilon = 10^{-6}$

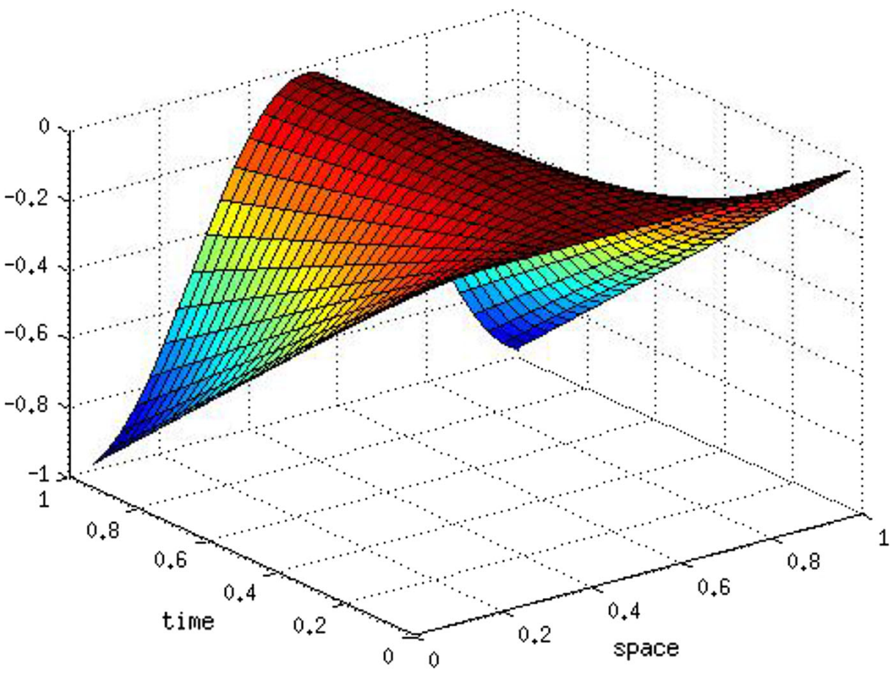

Fig. 6 Com sol for $a = 1 + xe^{-t}$ and $\epsilon = 10^{-6}$

Table 3 Sup norm and L^2-norm errors for $a = 1$ with data error $\delta = 0.1$

ϵ	Norm	$n = 1$	$n = 2$	$n = 3$	$n = 4$
10^{-2}	$\tilde{e}_\alpha$(Sup.)	1.0762e−02	2.6864e−03	2.7755e−03	2.7755e−03
	$\tilde{e}_\alpha$ (L^2)	1.5062e−01	1.9864e−02	2.5725e−02	2.5725e−02
10^{-4}	$\tilde{e}_\alpha$(Sup.)	1.1759e−02	2.1054e−03	2.3323e−03	2.3323e−03
	$\tilde{e}_\alpha$ (L^2)	1.6286e−01	1.3015e−02	1.4022e−02	1.4022e−02
10^{-6}	$\tilde{e}_\alpha$(Sup.)	1.4476e−02	1.9137e−03	7.7292e−04	6.3622e−04
	$\tilde{e}_\alpha$ (L^2)	1.9600e−01	2.1573e−02	7.4643e−03	6.5290e−03
10^{-8}	$\tilde{e}_\alpha$(Sup.)	1.4581e−02	1.8327e−03	8.8054e−04	7.5048e−04
	$\tilde{e}_\alpha$ (L^2)	1.9851e−01	2.2762e−02	8.9514e−03	7.6545e−03
10^{-10}	$\tilde{e}_\alpha$(Sup.)	1.4574e−02	1.7423e−03	8.4917e−04	7.8535e−04
	$\tilde{e}_\alpha$ (L^2)	1.9607e−01	2.0815e−02	7.4701e−03	6.6531e−03

Table 4 Sup norm and L^2-norm errors for $a = 1$ with data error $\delta = 0.01$

ϵ	Norm	$n = 1$	$n = 2$	$n = 3$	$n = 4$
10^{-2}	$\tilde{e}_\alpha$(Sup.)	2.7456e−03	2.7609e−03	2.7611e−03	2.7611e−03
	$\tilde{e}_\alpha$ (L^2)	2.6245e−02	2.8219e−02	2.8253e−02	2.8255e−02
10^{-4}	$\tilde{e}_\alpha$(Sup.)	1.0236e−02	2.0638e−03	2.2893e−03	2.2934e−03
	$\tilde{e}_\alpha$ (L^2)	1.4184e−01	1.1729e−02	1.3063e−02	1.3104e−02
10^{-6}	$\tilde{e}_\alpha$(Sup.)	2.8495e−03	1.4785e−04	8.5080e−05	7.3265e−05
	$\tilde{e}_\alpha$ (L^2)	3.9082e−02	1.6487e−03	7.3249e−04	6.2709e−04
10^{-8}	$\tilde{e}_\alpha$(Sup.)	3.1274e−03	1.6304e−04	8.4318e−05	7.7156e−05
	$\tilde{e}_\alpha$ (L^2)	4.2373e−02	1.8053e−03	7.2772e−04	7.2619e−04
10^{-10}	$\tilde{e}_\alpha$(Sup.)	3.1410e−03	1.6718e−04	7.2743e−05	6.8112e−05
	$\tilde{e}_\alpha$ (L^2)	4.2359e−02	1.9607e−03	6.9669e−04	6.4786e−04

Table 5 Sup norm and L^2-norm errors for $a = 1 + xe^{-t}$ with data error $\delta = 0.1$

ϵ	Norm	$n = 1$	$n = 2$	$n = 3$	$n = 4$
10^{-2}	$\tilde{e}_\alpha$(Sup.)	1.1059e−02	2.5699e−03	2.6622e−03	2.6622e−03
	$\tilde{e}_\alpha$ (L^2)	1.4722e−01	1.9717e−02	2.3479e−02	2.3479e−02
10^{-4}	$\tilde{e}_\alpha$(Sup.)	1.0397e−02	1.9212e−03	2.4031e−03	2.4031e−03
	$\tilde{e}_\alpha$ (L^2)	1.3420e−01	1.1214e−02	1.1374e−02	1.1502e−02
10^{-6}	$\tilde{e}_\alpha$(Sup.)	1.4579e−02	1.7309e−03	7.4623e−04	6.9256e−04
	$\tilde{e}_\alpha$ (L^2)	1.8966e−01	1.9290e−02	6.7521e−03	6.2181e−03
10^{-8}	$\tilde{e}_\alpha$(Sup.)	1.5510e−02	2.0304e−03	8.5458e−04	7.1499e−04
	$\tilde{e}_\alpha$ (L^2)	1.9076e−01	2.0370e−02	7.9749e−03	7.2298e−03
10^{-10}	$\tilde{e}_\alpha$(Sup.)	1.4927e−02	1.9658e−03	8.4008e−04	6.8129e−04
	$\tilde{e}_\alpha$ (L^2)	1.8983e−01	1.9640e−02	7.7436e−03	7.2644e−03

Table 6 Sup norm and L^2-norm errors for $a = 1 + xe^{-t}$ with data error $\delta = 0.01$

ϵ	Norm	$n = 1$	$n = 2$	$n = 3$	$n = 4$
10^{-2}	$\tilde{e}_\alpha$(Sup.)	2.7312e−03	2.7662e−03	2.7670e−03	2.7670e−03
	$\tilde{e}_\alpha$ (L^2)	2.2634e−02	2.6203e−02	2.6294e−02	2.6294e−02
10^{-4}	$\tilde{e}_\alpha$(Sup.)	8.8436e−03	1.9864e−03	2.2970e−03	2.3106e−03
	$\tilde{e}_\alpha$ (L^2)	1.1555e−01	8.8754e−03	9.8416e−03	9.8416e−03
10^{-6}	$\tilde{e}_\alpha$(Sup.)	2.9981e−03	1.6705e−04	7.8404e−05	7.4537e−05
	$\tilde{e}_\alpha$ (L^2)	3.7978e−02	1.5942e−03	7.7179e−04	7.7135e−04
10^{-8}	$\tilde{e}_\alpha$(Sup.)	3.2688e−03	1.9387e−04	7.7142e−05	7.2682e−05
	$\tilde{e}_\alpha$ (L^2)	4.1344e−02	1.8187e−03	7.4755e−04	7.3032e−04
10^{-10}	$\tilde{e}_\alpha$(Sup.)	3.2779e−03	1.5607e−04	6.4180e−05	5.8807e−05
	$\tilde{e}_\alpha$ (L^2)	4.1105e−02	1.6576e−03	5.8981e−04	5.9678e−04

be $g(\alpha) = \gamma\epsilon^2$ and $g(\alpha) = \gamma^2(\delta + \epsilon)^2$, where g is defined as in (4.5). The nonlinear equation involving g is solved by Newton method. We choose two different models for study, one with $a = 1$ and another with $a = 1 + xe^{-t}$. We divide this section in to three subsections. In the first subsection, we present the numerical result for different models with out data error in f and in the second subsection the corresponding result with perturbed data $\tilde{f}$. In final subsection, we compare the regularized scheme with other exiting schemes, such as, back word and Crank-Nicolson schemes with Shishkin mesh .

The $1D$ parabolic reaction-diffusion singularly perturbed PDE is defined by $Lu = f$ on Ω, that is

$$\frac{\partial u}{\partial t} - \epsilon \frac{\partial^2 u}{\partial x^2} + a(x,t)u = f(x,t) \text{ on } \Omega. \tag{5.1}$$

The Crank–Nicolson scheme is used to discretize perturbed parabolic PDE and the discrete version is the following

$$\begin{aligned} &b_1 u_{i-1,j} + b_2 u_{i-1,j+1} + b_3 u_{i,j} + b_4 u_{i,j+1} + b_5 u_{i+1,j} + b_6 u_{i+1,j+1} \\ &\quad = \frac{1}{2}(f(i,j) + f(i,j+1)) \end{aligned} \tag{5.2}$$

where

$$b_1 = b_2 = b_2 = b_5 = b_6 = \frac{-\epsilon}{2h^2},$$

$b_3 = \frac{-1}{k} + \frac{\epsilon}{h^2} + \frac{1}{2}a_{i,j}$ and $b_4 = \frac{1}{k} + \frac{\epsilon}{h^2} + \frac{1}{2}a_{i,j+1}$.
The spacing $x-$direction and $t-$direction denoted by h and k respectively. The numerical solution and data values are represented by $u_{i,j}$ and $f_{i,j}$ respectively. From the above system of equations we get the matrix L_{dis} and hence the iterative Tikhonov regularized solution will be

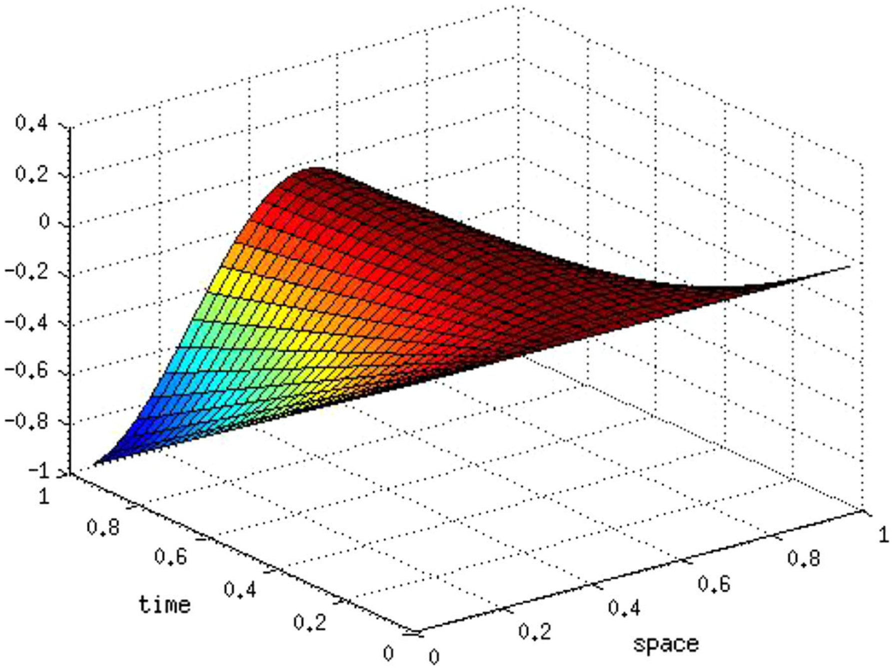

Fig. 7 Com. sol for $a = 1$; $\delta = 1$ %; $\epsilon = 10^{-8}$

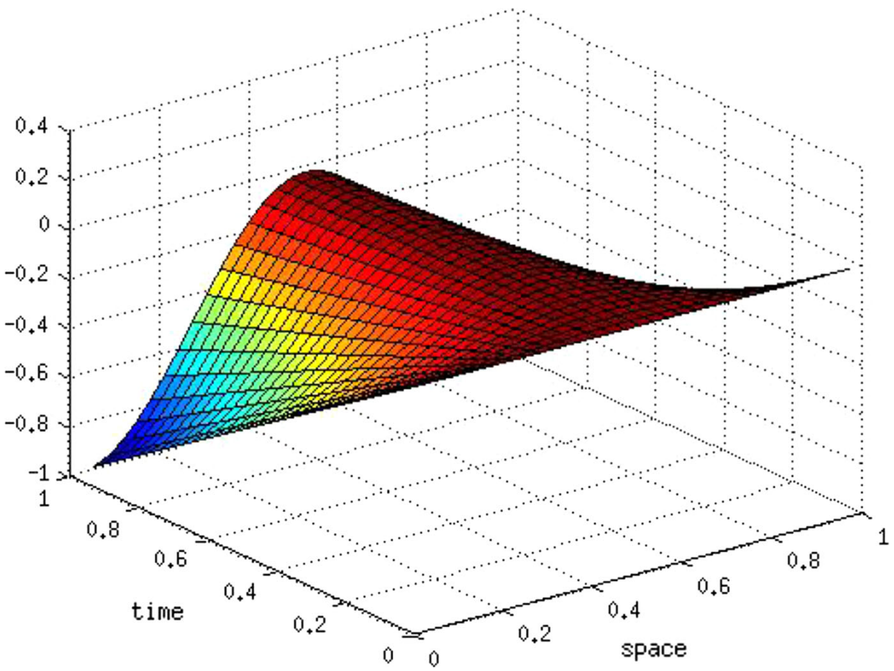

Fig. 8 Com. sol for $a = 1 + xe^{-t}$; $\delta = 1$ %; $\epsilon = 10^{-8}$

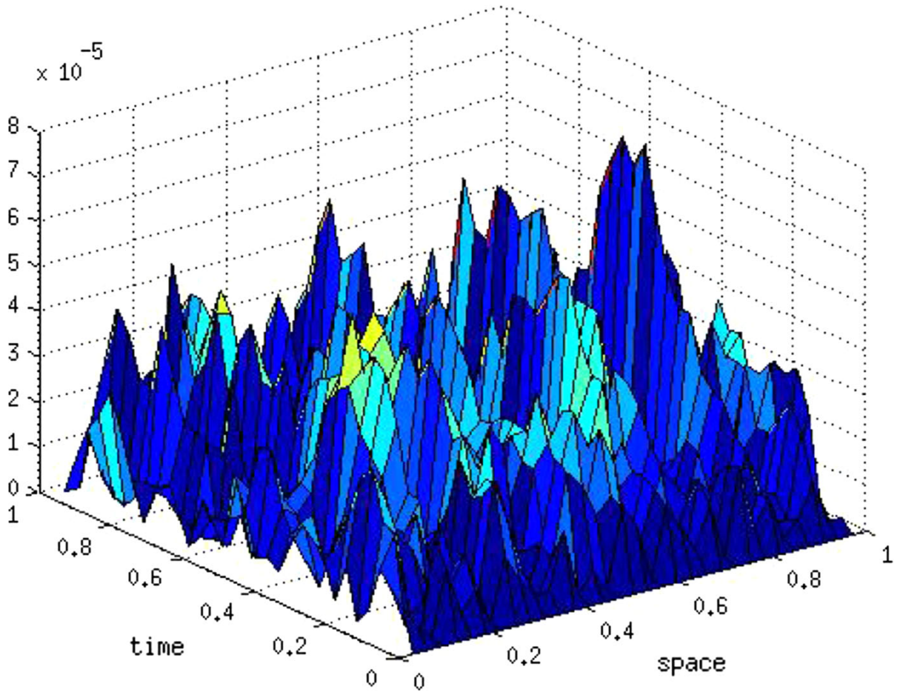

Fig. 9 Reg.error for $a = 1$; $\delta = 1$ %; $\epsilon = 10^{-8}$

$$u^n_{\alpha,dis} = \sum_{l=1}^{n} \alpha^{l-1} L^T_{dis}(L_{dis}L^T_{dis} + \alpha)^{-l} f_{dis}.$$

We represent the error $\|u - u_\alpha\|$ as e_α and $\|u - \tilde{u}_\alpha\|$ as $\tilde{e}_\alpha$ respectively.

5.1 $a = 1, a = 1 + xe^{-t}$ and no error in the data f

In this subsection, we see numerical solutions with no error in the data f of the model (1.1) with $a = 1$ and $a = 1 + xe^t$. The data f is taken such that exact solution of (1.1) is

$$u = t\left(\frac{e^{-x/\sqrt{\epsilon}} - e^{-(1-x)/\sqrt{\epsilon}}}{1 - e^{-1/\sqrt{\epsilon}}} - \cos(\pi x)^2\right).$$

Tables 1 and 2 show the numerical results of the model (1.1) with $a = 1$ and $a = 1 + xe^{-t}$ respectively on the grid size 32×32. We observe from these tables that the numerical result confirms the theoretical convergence rate of $O(\epsilon^{2/3})$. Figures 1 and 2 represent actual and computational solutions respectively and corresponding error is given by Fig. 3 for $a = 1$ with $\epsilon = 10^{-8}$. For $a = 1 + xe^{-t}$ with $\epsilon = 10^{-6}$, actual and computed solutions are presented in Figures 5 and 6 respectively and the

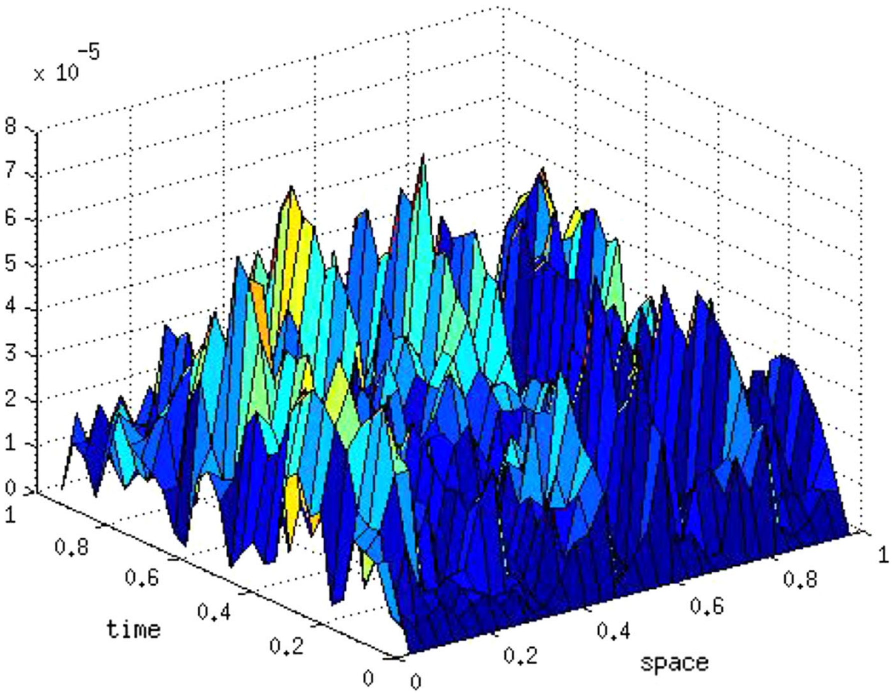

Fig. 10 Reg.error for $a = 1 + xe^{-t}$; $\delta = 1\ \%$; $\epsilon = 10^{-8}$

corresponding error is given in Fig. 4. We can easily see from the error figure that there is a boundary layer phenomena at edge $x = 0$ and $x = 1$. The parameter α is evaluated using the a posteriori parameter choice rule.

5.2 $a = 1$, $a = 1 + xe^{-t}$ and the data with data error in f

In this subsection, we discuss the case with model having inputs of perturbed data. In order to see how well regularized scheme works with data error, we introduced the random error in the data f and tried to obtain a stable approximate solution for the problem. We slightly perturbed the data f with 1 and 10 % random error for $a = 1$, $a = 1 + xe^{-t}$ and applied the regularized scheme. The computational results are summarized in Tables 3, 4, 5 and 6 for the model (1.1) with $a = 1$ and $a = 1 + xe^{-t}$ when $\delta = 1$ and 10 % respectively. The computational results assert the theoretical convergence rate $O((\epsilon + \delta)^{2/3})$. For $\delta = 1\ \%$, Figs. 7 and 8 display the computational solution of the model (1.1) with $a = 1$ and $a = 1 + xe^{-t}$ respectively and the corresponding errors are given in Figs. 9 and 10.

5.3 Comparison with other schemes

In this subsection, our idea is to study how good the approximate solution obtained through regularization technique compared with the standard schemes. We obtain the

Table 7 Sup norm and L^2-norm errors

ϵ	N	Norm	BW.error (Shishkin)		CN.error (Shishkin)		Reg.error	
			$a=1$	$a=1+xe^{-t}$	$a=1$	$a=1+xe^{-t}$	$a=1$	$a=1+xe^{-t}$
10^{-2}	2^5	*Sup.*	7.6481e−04	7.5931e−04	7.6150e−04	6.7356e−04	2.7696e−03	2.7652e−03
		L^2	7.2963e−03	6.9314e−03	7.1896e−03	5.7254e−03	2.8363e−02	2.6318e−02
	2^6	*Sup.*	1.9565e−04	1.9398e−04	1.1595e−04	3.7369e−04	2.8197e−03	2.8176e−03
		L^2	3.5976e−03	3.4214e−03	3.5710e−03	6.3023e−03	5.7359e−02	5.3619e−02
	2^7	*Sup.*	4.8984e−05	4.8584e−05	4.8922e−05	2.1697e−04	2.8390e−03	2.8379e−03
		L^2	1.7817e−03	1.6953e−03	1.7751e−03	7.2140e−03	1.1505e−02	1.0768e−01
10^{-4}	2^5	*Sup.*	1.0919e−02	1.0903e−02	1.0831e−02	1.0790e−02	2.3110e−03	2.2555e−03
		L^2	4.8705e−02	4.6232e−02	4.7943e−02	4.4465e−02	1.3230e−02	9.7344e−03
	2^6	*Sup.*	4.7284e−03	4.7244e−03	4.7164e−03	4.7054e−03	1.1068e−03	1.0896e−03
		L^2	3.6962e−02	3.5208e−02	3.6702e−02	3.5947e−02	1.2283e−02	1.0436e−02
	2^7	*Sup.*	1.6754e−03	1.6738e−03	1.6731e−03	1.6672e−03	9.2583e−04	1.1352e−03
		L^2	2.4396e−02	2.3279e−02	2.4314e−02	2.6860e−02	1.2860e−02	1.2644e−02
10^{-6}	2^5	*Sup.*	1.0919e−02	1.0917e−02	1.0832e−02	1.0828e−02	5.9579e−05	5.8307e−05
		L^2	4.8707e−02	4.6244e−02	4.7945e−02	4.5712e−02	3.4191e−04	2.5581e−04
	2^6	*Sup.*	4.7285e−03	4.7281e−03	4.7165e−03	4.7154e−03	2.3678e−04	2.3420e−04
		L^2	3.6963e−02	3.5216e−02	3.6703e−02	3.7826e−02	1.9202e−03	1.4456e−03
	2^7	*Sup.*	1.6754e−03	1.6753e−03	1.6732e−03	1.6726e−03	8.9896e−04	8.9399e−04
		L^2	2.4396e−02	2.3283e−02	2.4315e−02	2.9766e−02	1.0288e−02	7.7526e−03
10^{-8}	2^5	*Sup.*	1.0919e−02	1.0919e−02	1.0832e−02	1.0831e−02	5.9707e−07	5.8433e−07
		L^2	4.8707e−02	4.6245e−02	4.7945e−02	4.5818e−02	3.4270e−06	2.5647e−06
	2^6	*Sup.*	4.7285e−03	4.7284e−03	4.7165e−03	4.7164e−03	2.3887e−06	2.3628e−06
		L^2	3.6963e−02	3.5216e−02	3.6703e−02	3.7980e−02	1.9382e−05	1.4606e−05
	2^7	*Sup.*	1.6754e−03	1.6754e−03	1.6732e−03	1.6731e−03	9.5534e−06	9.5019e−06
		L^2	2.4396e−02	2.3284e−02	2.4315e−02	2.9993e−02	1.0960e−04	8.2893e−05
10^{-10}	2^5	*Sup.*	1.0919e−02	1.0919e−02	1.0832e−02	1.0832e−02	5.9708e−09	5.8435e−09
		L^2	4.8707e−02	4.6246e−02	4.7945e−02	4.5829e−02	3.4271e−08	2.5647e−08
	2^6	*Sup.*	4.7285e−03	4.7285e−03	4.7165e−03	4.7165e−03	2.3890e−08	2.3631e−08
		L^2	3.6963e−02	3.5216e−02	3.6703e−02	3.7995e−02	1.9384e−07	1.4608e−07
	2^7	*Sup.*	1.6754e−03	1.6754e−03	1.6732e−03	1.6731e−03	9.5569e−08	9.5048e−08
		L^2	2.4396e−02	2.3284e−02	2.4315e−02	3.0015e−02	1.0964e−06	8.2924e−07

numerical solutions of the model (1.1) with $a = 1$ and $a = 1 + xe^{-t}$ through Back ward and Crank–Nicolson schemes on Shishkin mesh for different values of ϵ and mesh sizes. The numerical results of three schemes are provided in Table 7 for $a = 1$ and $a = 1 + xe^{-t}$. For $a = 1$ and $\epsilon = 10^{-8}$, the numerical solution of Back ward scheme and Crank–Nicolson schemes are showed in the Figs. 11 and 12 on the grid size 32×32 and their corresponding errors in the solutions are presented in Figs. 13, 14. Figures 15, 16 provide numerical solutions of Back ward Crank–Nicolson schemes

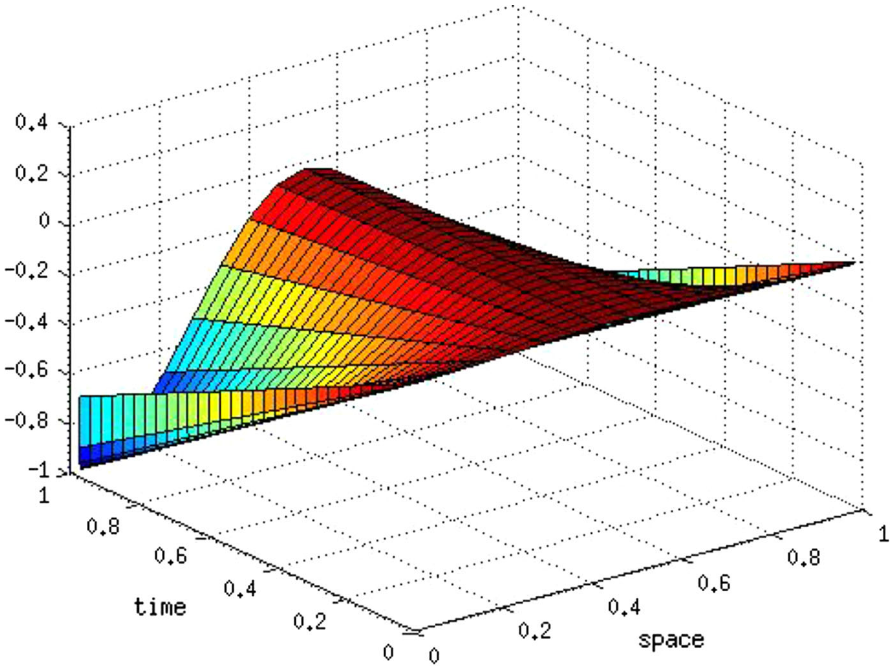

Fig. 11 BW.comp sol for $a = 1$, $\epsilon = 10^{-8}$

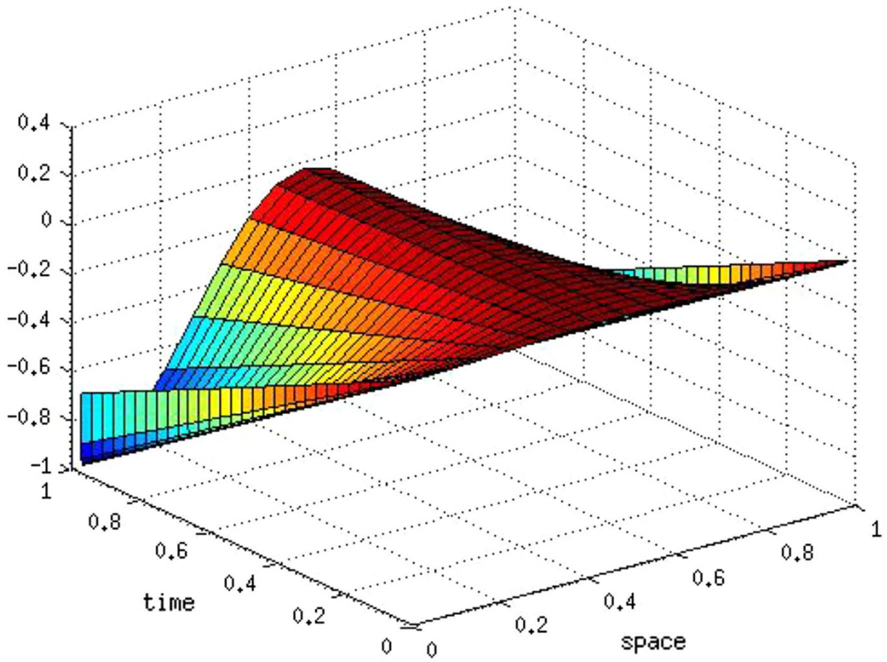

Fig. 12 CN.comp sol for $a = 1$, $\epsilon = 10^{-8}$

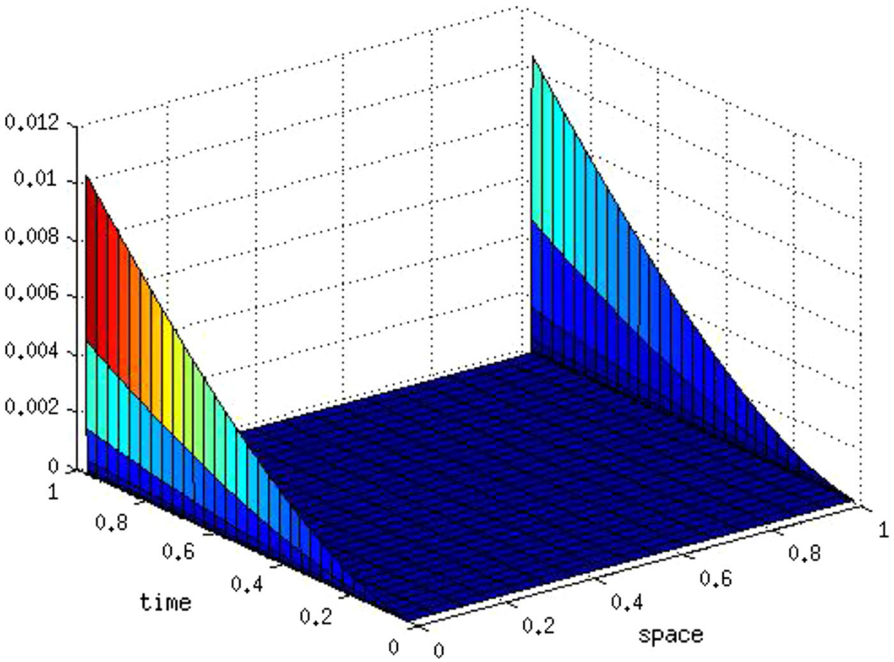

Fig. 13 BW.error for $a = 1, \epsilon = 10^{-8}$

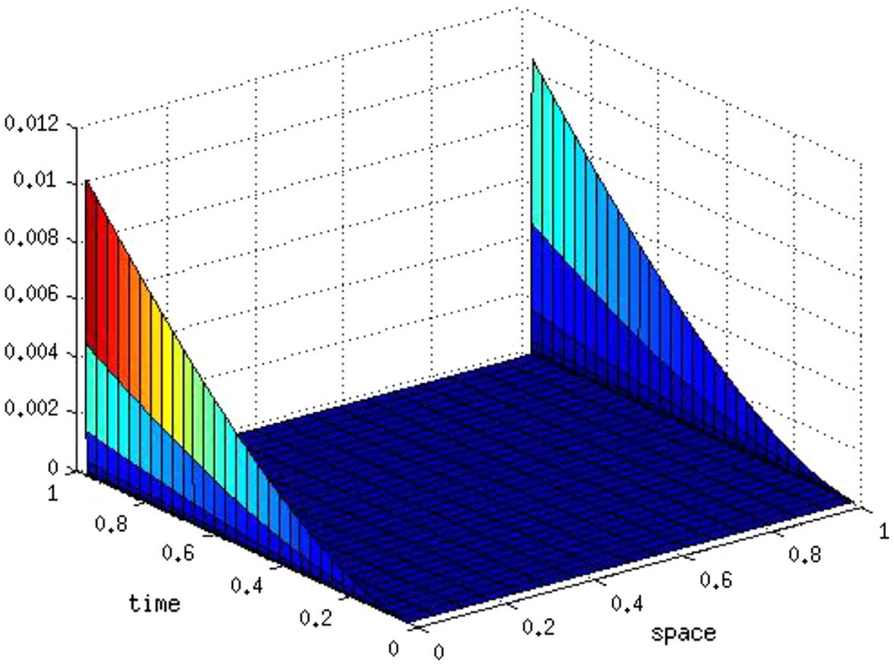

Fig. 14 CN.error for $a = 1, \epsilon = 10^{-8}$

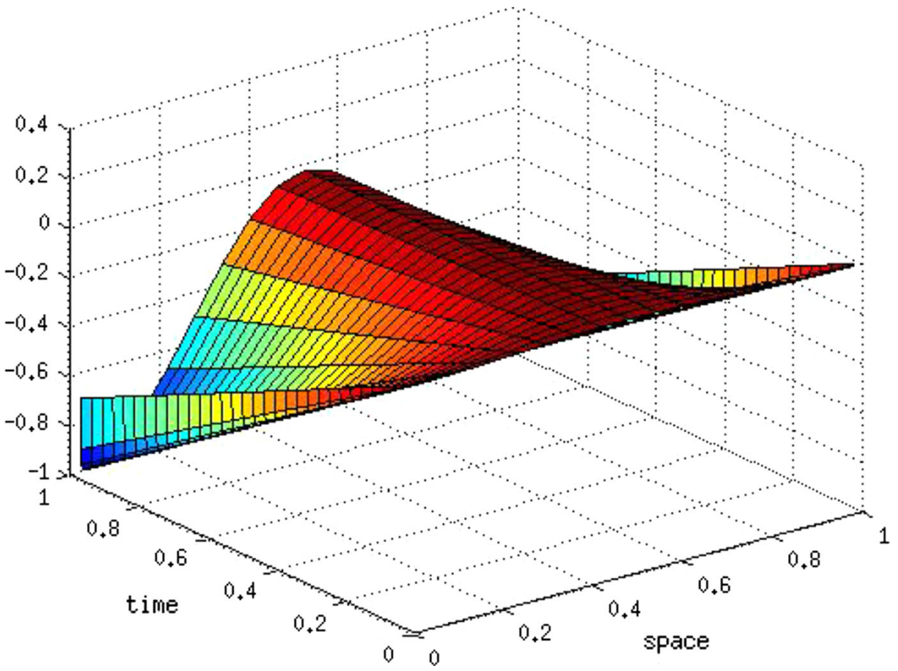

Fig. 15 BW.comp sol for $a = 1 + xe^{-t}$, $\epsilon = 10^{-6}$

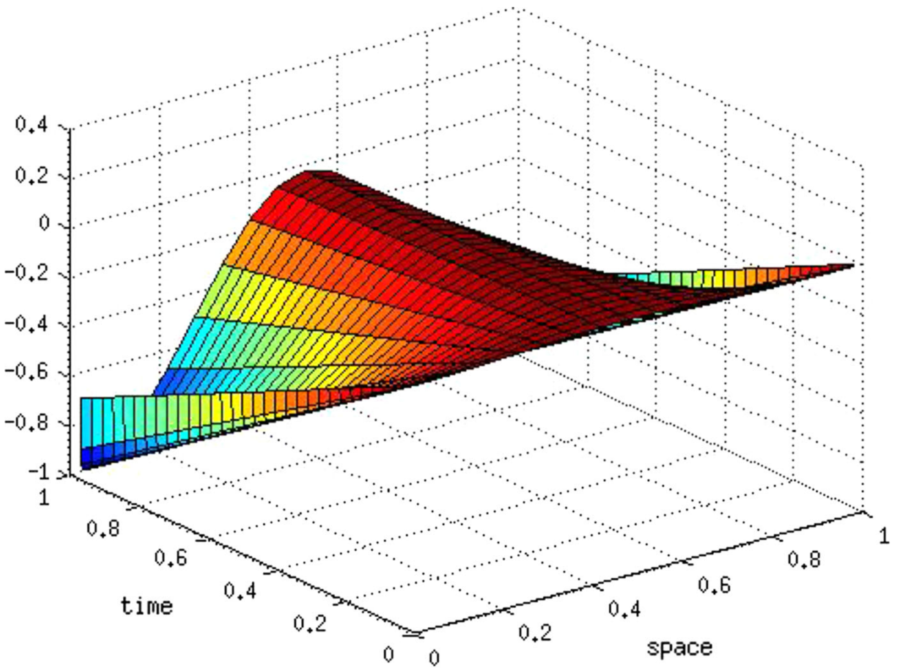

Fig. 16 CN.comp sol for $a = 1 + xe^{-t}$, $\epsilon = 10^{-6}$

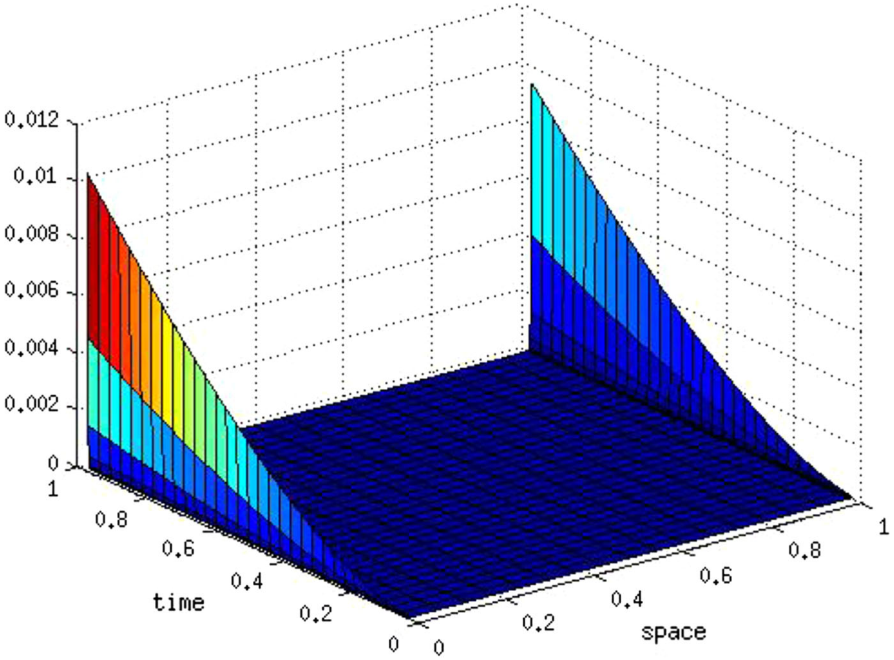

Fig. 17 BW.error for $a = 1 + xe^{-t}$, $\epsilon = 10^{-6}$

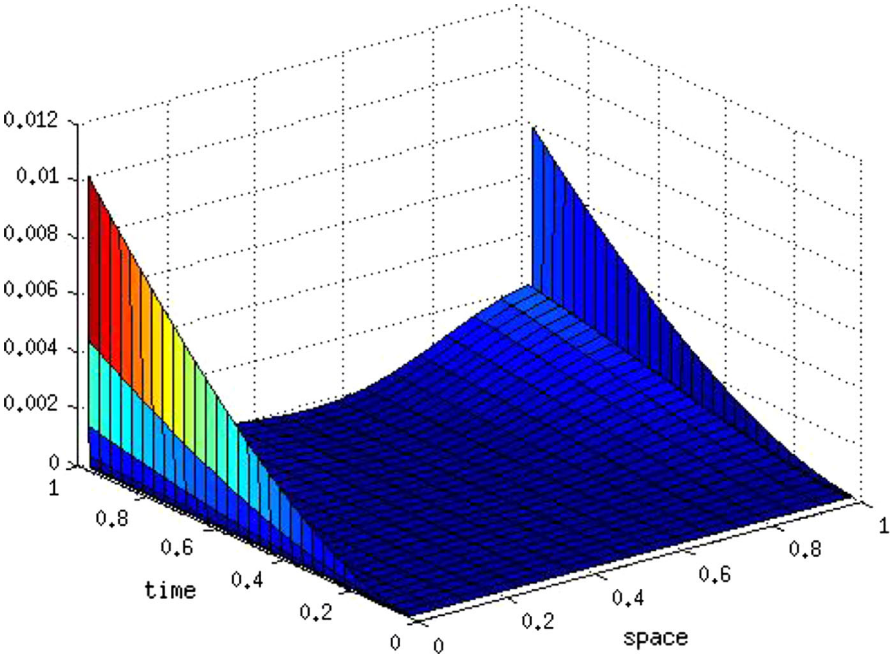

Fig. 18 CN.error for $a = 1 + xe^{-t}$, $\epsilon = 10^{-6}$

respectively when $a = 1 + xe^{-t}$ and $\epsilon = 10^{-6}$. The corresponding errors given in Figs. 17 and 18. The regularization parameter α obtained by an a priori parameter choice rule. The computational result indicates that iterative Tikhonov regularization can be considered as an alternative method for finding the solution for singularly perturbed parabolic problems.

Acknowledgments The authors would like to thank the unknown referees for their careful reading of the manuscripts and valuable suggestions.

References

1. Boglaev, I.P.: Finite difference domain decomposition algorithms for a parabolic problem with boundary layers. Comput. Math. Appl. **36**, 25–40 (1998)
2. Clavero, C., Gracia, J.L.: High order methods for elliptic and time dependent reaction-diffusion singularly perturbed problems. Appl. Math. Comput. **168**, 1109–1127 (2005)
3. Clavero, C., Gracia, J.L.: On the uniform convergence of a finite difference scheme for time dependent singularly perturbed reaction-diffusion problems. Appl. Math. Comput. **216**, 1478–1488 (2010)
4. Engl, H.W., Hanke, M., Neubauer, A.: Regularization of Inverse Problems. Kluwer, Dordrecht (1996)
5. George, S., Nair, M.T.: An apostoriori parameter choice for simpli- fied regularization of ill-posed problems. Integral Equ. Oper. Theory **16**, 392–399 (1993)
6. Gfrerer, H.: An a posteriori parameter choice for ordinary and iterated Tikhonov regularization of ill-posed problems leading to optimal convergence rates. Math. Comp. **49**(180), 507–522 (1987)
7. Groetsch, C.W.: Stable Approximate Evaluation of Unbounded Operators. Springer, Heidelberg (2007)
8. Hemker, P.W., Shishkin, G.I., Shishkina, L.P.: ϵ-uniform schemes with high-order time accuracy for parabolic singular perturbation problems. IMA J. Numer. Anal. **20**, 99–121 (2000)
9. Kadalbajoo, M., Gupta, V.: A brief survey on numerical methods for solving singularly perturbed problems. Appl. Math. Comput. **217**, 3641–3716 (2010)
10. Kopteva, N.: On the uniform in small parameter convergence of a weighted scheme for the one dimensional time dependent convection-diffusion equation. Comput. Math. Math. Phys. **37**, 1173–1180 (1997)
11. Linß, T., Madden, N.: Parameter uniform approximations for time-dependent reaction-diffusion problems. Numer. Methods Partial Differ. Equ. **23**, 1290–1300 (2007)
12. Mukherjee, K., Natesan, S.: Richardson extrapolation technique for singularly perturbed parabolic convection-diffusion problems. Computing **92**, 132 (2011)
13. Ng-Stynes, M.J., O'Riordan, E., Stynes, M.: Numerical methods for time-dependent convection-diffusion equations. J. Comput. Appl. Math. **21**, 289–310 (1988)
14. Rajan, M.P., Reddy, G.D. : An Iterative Tikhonov regularization for solving singularly perturbed elliptic PDE (Under review)
15. Rajan, M.P.: Convergence analysis of a regularized approximation for solving Fredholm integral equations of the first kind. J. Math. Anal. Appl **279**, 522–530 (2003)
16. Ross, H.-G., Stynes, M., Tobiska, L.: Robust Numerical Methods for Singularly Perturbed Elliptic Differential Equations. Springer-Verlag, Berlin (2008)
17. Sheela, S., Singh, A.: Tikhonov regularization of an elliptic PDE. Math. Comput. Simul. **57**, 1–4 (2001)
18. Shishkin, G.I. : Grid approximation of singularly perturbed elliptic and parabolic equations, Second doctoral thesis, Keldysh Institute, Moscow (1991)
19. Shishkin, G.I.: On finite difference fitted schemes for singularly perturbed boundary value problems with a parabolic boundary layer. J. Math. Anal. Appl. **208**, 181–204 (1997)